B: Some Calculus Background for Partial Differential Equations

It is assumed that anyone embarking on the study of partial differential equations has at their fingertips the basics of partial differentiation and integration of (multivariable) functions, and elementary (ordinary) differential equations. If there has been too much of a gap since you took those courses, you need to spend time reviewing that material, or you will not be successful in learning partial differential equation techniques. Below is a brief guide to some needed calculus material, but it is by no means a complete representation of all relevant material.

1 Integration and Differentiation

Here is a result concerning an integral $I(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$, which has variable limits of integration, that is needed for the subject, and various exercises:

Leibniz Theorem: If $f(x,t)$ and $\partial f/\partial t$ are continuous on the rectangle $[A,B] \times [c,d]$, where $[A,B]$ contains the union of all intervals $[a(t), b(t)]$, and if $a(t)$ and $b(t)$ are differentiable on $[c,d]$, then

$$\frac{dI}{dt} = \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t),t) - f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx\ .$$

Exercises

1. Let $I(t) = \int_t^{t^2} \sin(x)\,dx$. First, use the Leibniz theorem to compute dI/dt. Second, integrate the integral directly, then take the derivative to obtain the same result.

2. Define the two-variable function $u(x,t) = \int_{x-t}^{x+t} g(y)\,dy$ for an arbitrary integrable function $g(y)$, and show that $u(x,t)$ is a solution to the partial differential equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$.

Now we consider some operators from multivariable calculus.

1. *grad*: Let $\phi = \phi(x,y,z)$ be a differentiable function. Then

$$grad\ \phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\ ,$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the unit vectors in the x, y, z directions, respectively. Notationally, *grad* ϕ is also written as $\nabla\phi$.

2. *div*: Suppose $\mathbf{F} = (f_1, f_2, f_3)$ is a vector function from $\mathbb{R}^3$ to $\mathbb{R}^3$, then

$$div\ \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\ .$$

3. *curl*:

$$curl\ \mathbf{F} = \nabla \times \mathbf{F} = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})\ .$$

If you remember how to take the determinant of a 3×3 matrix, one way of remembering the *curl* is by

$$\nabla \times \mathbf{F} = det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}\ .$$

These can be combined in various ways. An important case is

$$div\ grad = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\ .$$

If $u = u(x, y, z)$ has two derivatives in each variable, then

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called **Laplace's equation**.

Now consider domain $D \subset \mathbb{R}^n$, where usually $n = 2, 3$. By domain in these Notes we mean an open, bounded, connected set, which is also simply connected ("no holes"). We also assume D has a *piecewise smooth* boundary, denoted by ∂D. By this we mean it has a unique unit outward normal vector, ν, defined everywhere on ∂D, except possibly at a finite number of points (think of a rectangle in the plane), or a lower dimensional set for $n \geq 3$, like a cube. Then we have the important theorem for our use

Divergence Theorem: Let g be any continuous scalar function defined in D and its boundary ∂D, and be continuously differentiable in D, and let F be a continuously differentiable vector function defined in D. Then

$$\int_D \{g\ div\ (F) + F \cdot grad\ (g)\}\ dx = \int_{\partial D} g\ F \cdot \nu ds\ \ .$$

Remark about notation: This theorem and others we mention in these Notes hold in arbitrary dimension, so instead of writing multiple integral signs, the usual practice is to use one integral sign and let the context determine the details. For example, in the plane, if $D = \{(x, y) : |x| < 1, 0 < y < 3\}$, instead of writing $\int_0^3 \int_{-1}^1 f(x, y)\ dxdy$, we might write $\int_D f(\text{x})\ d\mathbf{x}$, assuming D has been previously defined, and assuming $\mathbf{x} = (x, y)$. The differential ds on the right side of the above expression has to be defined in terms of the geometry of the boundary.

Exercises

1. Let $F = \nabla u = grad(u)$, and substitute this into the divergence theorem to obtain $\int_D \{g\nabla^2 u + \nabla u \cdot \nabla g\}dx = \int_{\partial D} g(\nabla u \cdot \nu)ds$. This is *Green's first identity.*

2. (Green's second identity): Take Green's first identity, exchange g and u, and subtract to obtain
$\int_D \{g\nabla^2 u + \nabla u \cdot \nabla g\}dx = \int_{\partial D}(u\nabla g - g\nabla u) \cdot \nu ds$. This is *Green's second identity.*

These results hold in arbitrary dimensions $n \geq 2$, but a useful theorem for just the planar domains is

Green's Theorem: Let D be a planar domain with the characteristics mentioned above, Consider ∂D parameterized such that it is traversed once with D on the left (traversed counterclockwise). Let $p(x, y)$ and $q(x, y)$ be continuously differentiable functions defined on the closure of D, i.e. on D and its boundary. Then

$$\int_D (q_x - p_y)\ dxdy = \int_{\partial D} pdx + qdy$$

2 Trigonometric and Hyperbolic Functions

We use the trig. addition formulas over and over in this class, so

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

In particular, $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$, so, for example, $\sin^2(x) = (1 - \cos(2x))/2$.

Exercises

1. For arbitrary positive integers n, m, show

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

$$\int_0^1 \cos(n\pi x)\sin(m\pi x)\,dx = 0 \quad \text{for all } n \text{ and } m$$

2. There are any number of sources showing the graphs of the trig functions, so you are responsible for knowing the graphs of the trig functions. Sketch a graph of $\tan(x)$ for $x > 0$ and superimpose on the graph the graph of $x/2$. Numerically approximate the first 5 positive solutions to the transcendental equation $\tan(x) = x/2$.

3. Show that $\sin(11\pi x)\cos(10\pi x) = (\sin(21\pi x) + \sin(\pi x))/2$.

The *hyperbolic functions* are $sinh(x) = (e^x - e^{-x})/2, cosh(x) = (e^x + e^{-x})/2, tanh(x) = sinh(x)/cosh(x)$, etc., so you should be able to sketch their graphs and know that

$$\frac{d}{dx}\sinh(x) = \cosh(x) \quad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

Exercises

1. Show that $\sinh(x), \tanh(x)$ are odd functions and sketch a graph of each. Show that $\cosh(x), sech(x)$ are even functions and sketch a graph of each.

2. Show that $\sinh(ax)$ and $\cosh(ax)$ form a fundamental set of solutions for the differential equation $\frac{d^2u}{dx^2} - a^2u = 0$. So, you must show they satisfy the equation and that they are linearly independent.

3 Sequences and Series of Functions

We will be dealing with series of functions in these Notes, so you should recall a few properties about sequences and series.

Definition: ***Convergence of a series***: A series of real numbers $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$ converges if the tail end can be made arbitrarily small, i.e. given any tolerance $\varepsilon > 0$, there is an $M > 1$ such that for $m \geq M$, $|\sum_{n=m}^{\infty} a_n| < \varepsilon$.

Definition: ***Absolute Convergence***: Series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Remark on ***Comparison Test****:* If $|a_n| \leq b_n$ for all n, and if $\sum_1^{\infty} b_n$ converges, then $\sum_1^{\infty} a_n$ converges absolutely. The countrapositive necessarily follows: If $\sum_1^{\infty} |a_n|$ diverges, so does $\sum_1^{\infty} b_n$. The **limit comparison test** states that if $a_n \geq 0, b_n \geq 0$, if $\lim_{n\to\infty} a_n/b_n = L$, where $0 \leq L < \infty$, and if $\sum_1^{\infty} b_n$ converges, then so does $\sum_1^{\infty} a_n$.

Remark on the ***Ratio Test****:* The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq \rho < 1$, for $n \geq N \geq 1$. (We do not care if the inequality is not met for the first N terms.)

Examples: For $\sum_1^{\infty} (\frac{1}{2})^n$, that is, $a_n = 1/2^n$, hence $|a_{n+1}|/|a_n| = 1/2$, so the series converges absolutely. For $\sum_1^{\infty} \frac{1}{n}$, $|a_{n+1}|/|a_n| = \frac{n}{n+1} \to 1$. Hence, there is no upper bound less than 1, so the ratio test fails, i.e. gives us no information. This series actually diverges. The ratio test also fails for the series $\sum_1^{\infty} \frac{1}{n^2}$, but the series converges to $\pi^2/6$. In fact, the p-series $\sum_1^{\infty} \frac{1}{n^p}$ converges for any $p > 1$, and diverges (is infinite) for $p \leq 1$.

Definition: ***Uniform convergence of a sequence of functions***: Assume the sequence $\{f_x\}_{n=1,2,\ldots}$ of functions is defined on an interval $\mathcal{I}$ of $\mathbb{R}$. Then $\{f_x\}_{n=1,2,\ldots}$ converges uniformly on $\mathcal{I}$ to $f(x)$ if for any tolerance $\varepsilon > 0$, there is an M such that for $m > M$, $|f_m(x) - f(x)| < \varepsilon$ for all $x \in \mathcal{I}$.

Definition: ***Uniform convergence of a series of functions***: If the $f_n's$ are defined on an interval $\mathcal{I}$, then $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly** on $\mathcal{I}$ to $f(x)$ if the *sequence of partial sums* $\{S_N\}_{N\geq 1}$, where $S_N = \sum_{n=1}^{N} f_n(x)$, converges uniformly to $f(x)$ on $\mathcal{I}$.

Comparison Test: If $|f_n(x)| \leq c_n$, for all n and for all $x \in [a,b]$, where the $c_n's$ are constants, and if $\sum_1^{\infty} c_n$ converges, then $\sum_1^{\infty} f_n(x)$ converges *uniformly* in the interval $[a,b]$, as well as absolutely.

Convergence Theorem: If $\sum_1^{\infty} f_n(x)$ converges uniformly to $f(x)$ in the interval $[a,b]$, and if all the functions $f_n(x)$ are continuous in $[a,b]$, then the sum $f(x)$ is also continuous in $[a,b]$, and

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx \ .$$

This last statement is called term-by-term integration.

Convergence of Derivatives: If all the functions $f_n(x)$ are differentiable in $[a,b]$, and if the series $\sum_1^{\infty} f_n(c)$ converges for some c, and if the series of derivatives $\sum_1^{\infty} f_n'(x)$ converges *uniformly* in $[a,b]$, then $\sum_1^{\infty} f_n(x)$ converges uniformly to a function $f(x)$ and

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x) \quad \text{for } x \in [a,b] \ .$$

This is term-by-term differentiation.